21/11/2019

$k =$	0	1	2	3	4	5
$n = 1$	1					
$n = 2$	1	1				
$n = 3$	2	3	1	0	0	0
$n = 4$	6	11	6	1	0	0
$n = 5$	24	50	35	10	1	0

$\boxed{P_{nk}}$

$$P_{nk} = (n-1)\,P_{(n-1)k} + P_{(n-1)(k-1)}$$

$$P_{40} = 3 \cdot P_{30} + \underset{\overset{\shortparallel}{0}}{P_{3(-1)}} = 2 \cdot 3 + 0$$

$$P_{41} = 3 \cdot P_{31} + P_{30} = 3 \cdot 3 + 2 = 9 + 2 = 11$$

$$P_{42} = 3 \cdot P_{32} + P_{31} = 3 \cdot 1 + 3 = 6$$

$$1 \cdot (1+z)(2+z) = 2 + z + 2z + z^2 = \boxed{2} + \boxed{3}z + \boxed{1}z^2$$

$$4!\, G_4(z) = (1+z)(2+z)(3+z) = (2+3z+z^2)(3+z) = 6 + 2z + 9z + 3z^2 + 3z^2 + z^3$$

$$= \boxed{6} + \boxed{11}z + \boxed{6}z^2 + \boxed{1}z^3$$

$$G_n(z) := \frac{1 \cdot (1+z)(2+z) \cdots (n-1+z)}{n!}$$ è un polinomio di grado $n-1$

$$= \underline{P_{n0}} + \underline{P_{n1}z} + \underline{P_{n2}z^2} + \dots + \underline{P_{n(n-1)}}\, z^{n-1}$$

$$G_n(1) = \boxed{P_{n0}} + P_{n1} + P_{n2} + \dots + P_{n(n-1)} = 1$$

$$G_n'(z) = 0 + P_{n1}\, 1 + P_{n2}\, 2z + \dots + P_{n(n-1)}\,(n-1)z^{n-2}$$

$$G_n'(1) = P_{n1} \cdot 1 + P_{n2}\, 2 + P_{n3} \cdot 3 + \dots + P_{n(n-1)}\,(n-1)$$

$$= E\,A$$

Distribuzione di probabilità $p_0, p_1, p_2, p_3, p_4, \ldots$

Definito $G(z) = \sum_{k \geq 0} p_k z^k$, come abbiamo appena visto

$$G(1) = 0, \qquad G'(1) = \text{ave } G := \sum_{k \geq 0} k \, p_k$$

Supponiamo di avere un'altra distribuzione $q_0, q_1, q_2, q_3, q_4, \ldots$

Definiamo

$$H(z) = \sum_{k \geq 0} q_k z^k, \quad H(1) = 1 \qquad H'(1) = \text{ave } H$$

Considero adesso

$$K(z) = G(z) \cdot H(z), \text{ allora } K(1) = G(1) \cdot H(1) = 1 \cdot 1 = 1$$

quanto fa ave $K(z)$? $\quad \text{ave } K(z) = \big(G(z) \cdot H(z)\big)'(1)$

$$= G'(1) \cdot \underbrace{H(1)}_{=1} + \underbrace{G(1)}_{=1} \cdot H'(1)$$

$$= G'(1) + H'(1) = \text{ave } G + \text{ave } H$$

$$G_n(z) = \frac{1 (1+z)(2+z) \cdots (n-1+z)}{n!} \qquad n! = 2 \cdot 3 \cdot 4 \cdots n$$

$$= \frac{1+z}{2} \; \frac{2+z}{3} \cdots \frac{(n-1)+z}{n}$$

$$= h_n^1(z) \cdot h_n^2(z) \cdots h_n^{(n-1)}(z)$$

$$h_n^k(z) = \frac{k+z}{(k+1)} = \frac{k}{k+1} + \frac{1}{k+1} z$$

$$h_n^k(1) = \frac{k+1}{k+1} = 1$$

$$\text{ave}\, G_n = \text{ave}\, h_n^1 + \text{ave}\, h_n^2 + \dots + \text{ave}\, h_n^{(n-1)}$$

$$\text{ave}\, h_n^k = (h_n^k)'(1) = \frac{1}{k+1}$$

$$\text{ave}\, G_n = \sum_{k=1}^{n-1} \frac{1}{k+1} = \sum_{k=1}^{n} \frac{1}{k} - 1 = H_n - 1$$

numeri armonici

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$$

$$H_n \approx -\gamma + \log n$$

$$n = 1\,000\,000 \qquad \log n \approx 13$$

Numeri di Fibonacci

$$\begin{cases} F_0 = 0, \ F_1 = 1 \\ F_n = F_{n-1} + F_{n-2} \end{cases}$$

$$F_n = \frac{1}{\sqrt{5}} \left(\phi^n - \hat{\phi}^n \right) \qquad \phi = \frac{1+\sqrt{5}}{2} \qquad \hat{\phi} = \frac{1-\sqrt{5}}{2}$$

Definiamo

$$F(z) = F_0 + F_1 z + F_2 z^2 + F_3 z^3 + \dots = \sum_{k \geq 0} F_k z^k$$

$$\begin{aligned} + \quad F(z) &= F_0 + F_1 z + F_2 z^2 + F_3 z^3 + \dots \\ - \ z F(z) &= \qquad F_0 z + F_1 z^2 + F_2 z^3 + \dots \\ - z^2 F(z) &= \qquad\qquad F_0 z^2 + F_1 z^3 + \dots \end{aligned}$$

$$(1 - z - z^2) F(z) = \underset{\substack{\shortparallel \\ 0}}{F_0} + \underset{\substack{\shortparallel \\ 1}}{(F_1 - F_0)} z + (F_2 - F_1 - F_0) z^2 + (F_3 - F_2 - F_1) z^3 + \dots$$

$$F_2 = F_1 + F_0 \implies F_2 - F_1 - F_0 = 0$$

$$F_3 - F_2 - F_1 = 0$$

$$F_4 - F_3 - F_2 = 0$$

$$\vdots$$

$$(1 - z - z^2) F(z) = z$$

$$F(z) = \frac{z}{1 - z - z^2}$$

Funzione Generatrice dei numeri di Fibonacci.

$|\alpha| < 1$

$$\sum \alpha^k = \frac{1}{1-\alpha}$$

Sappiamo che $\displaystyle \frac{1}{1 - \alpha z} = \sum_{k \geq 0} (\alpha z)^k = \sum_{k \geq 0} \alpha^k z^k$

C'è speranza di esprimere $\displaystyle \frac{z}{1 - z - z^2}$ in termini di $\displaystyle \frac{1}{1 - \alpha z}$??

con opportune scelte di α ?

Mi posso chiedere $\exists$ A, B, α, β reali tali che

$$\frac{z}{1-z-z^2} = \frac{A}{1-\alpha z} + \frac{B}{1-\beta z} \; ?$$

$$= \frac{A(1-\beta z) + B(1-\alpha z)}{(1-\alpha z)(1-\beta z)}$$

$$\cancel{\exists\, \alpha, \beta \in \mathbb{R} \text{ tali che } 1-z-z^2 = (1-\alpha z)(1-\beta z)}$$

$$\begin{cases} A(1-\beta z) + B(1-\alpha z) = z & \text{uguaglianza numeratori} \\ (1-\alpha z)(1-\beta z) = 1-z-z^2 \quad (*) & \text{uguaglianza dei denominatori} \end{cases}$$

Considero $w^2 - wz - z^2$

$= (w - w_1)(w - w_2)$ dove w_1 e w_2 sono le radici di $w^2 - wz - z^2$

$$w_{12} = \frac{z \pm \sqrt{z^2 + 4z^2}}{2} = z\,\frac{1 \pm \sqrt{5}}{2}$$

$$w^2 - wz - z^2 = \left(w - \frac{1+\sqrt{5}}{2} z\right)\left(w - \frac{1-\sqrt{5}}{2} z\right)$$

Pongo $w = 1$

$$1 - z - z^2 = \left(1 - \frac{1+\sqrt{5}}{2} z\right)\left(1 - \frac{1-\sqrt{5}}{2} z\right)$$

$$= (1 - \phi z)(1 - \hat{\phi} z) \qquad (**)$$

$$\phi := \frac{1+\sqrt{5}}{2}, \quad \hat{\phi} := \frac{1-\sqrt{5}}{2}$$

$\alpha = \phi$, $\beta = \hat{\phi}$ confrontando $(*)$ con $(**)$.

$$A(\cancel{\alpha}\,1-\beta z)+B(1-\alpha z)=z$$

Se pongo $z=0$ ho che $A+B=0$, cioè $B=-A$

$$A\left(\cancel{1}-\beta z\ \cancel{-1}+\alpha z\right)=z \Rightarrow A\cancel{z}(\alpha-\beta)=\cancel{z}$$

$$A(\alpha-\beta)=1 \qquad \alpha-\beta=\phi-\hat{\phi}=\frac{1+\sqrt{5}}{2}-\frac{1-\sqrt{5}}{2}$$

$$=\frac{1+\sqrt{5}-1+\sqrt{5}}{2}=\sqrt{5}$$

Alla fine $A=-B=\frac{1}{\sqrt{5}}$

Dunque

$$\frac{z}{1-z-z^2}=\frac{1}{\sqrt{5}}\left(\frac{1}{1-\phi z}-\frac{1}{1-\hat{\phi}z}\right)$$

$$=\frac{1}{\sqrt{5}}\left(\sum_{k\geq 0}\phi^n z^n-\sum_{k\geq 0}\hat{\phi}^n z^n\right)$$

$$=\frac{1}{\sqrt{5}}\sum_{k\geq 0}\left(\phi^n-\hat{\phi}^n\right)z^n=\sum_{k\geq 0}\underbrace{\frac{1}{\sqrt{5}}\left(\phi^n-\hat{\phi}^n\right)}_{F_n}z^n$$

Quindi $F_n=\frac{1}{\sqrt{5}}\left(\phi^n-\hat{\phi}^n\right)$